

## UNSTEADY FLOW IN OPEN CHANNELS

### 10.1 Introduction

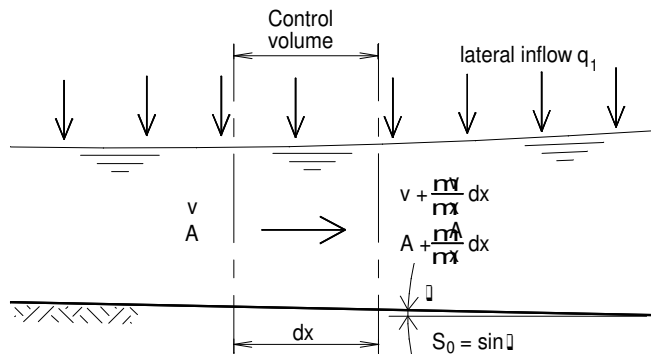
Unsteady flow in open channels differs from that in closed conduits in that the existence of a free surface allows the flow cross-section to freely change, a factor which has an important influence on the rate of transient change propagation. Unsteady open channel flow is encountered in flood flow in rivers, in headrace canals supplying hydropower stations, in river estuaries, and so on.

### 10.2 Basic equations

As in the case of closed conduits the basic equations are derived from continuity and momentum considerations. In deriving these equations the following assumptions are made:

1. Hydrostatic pressure prevails at every point in the channel.
2. Velocity is uniformly distributed over each cross-section.
3. The slope of the channel bed is small and uniform.
4. The frictional resistance is the same as for steady flow.

Fig 10.1 defines a control volume and the dimensional parameters used to develop the continuity equation.



**Fig 10.1 Reference diagram for the continuity equation**

The continuity equation balances mass inflow and mass outflow with the rate of change of the contained mass within the control volume:

In time  $dt$ : Inflow - Outflow = Change in mass of fluid in CV

$$\rho v A dt + \rho q_1 dt dx - \rho \left( v + \frac{\partial v}{\partial x} dx \right) \left( A + \frac{\partial A}{\partial x} dx \right) dt = \rho \frac{\partial A}{\partial t} dt dx$$

Dividing across by  $\rho \, dt \, dx$  and neglecting higher order terms:

$$A \frac{\partial v}{\partial x} + v \frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} = q_1 \quad (10.1)$$

Equation (10.1) can also be written in the form

$$\frac{\partial}{\partial x} (vA) + \frac{\partial A}{\partial t} = q_1$$

or

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q_1 \quad (10.2)$$

For a rectangular channel with zero lateral inflow, equation (10.1) simplifies to

$$y \frac{\partial v}{\partial x} + v \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0 \quad (10.3)$$

Hence

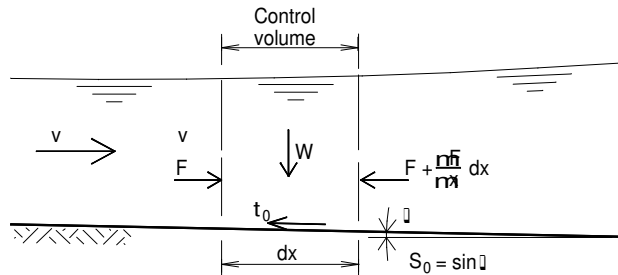
$$\frac{\partial}{\partial x} (vy) + \frac{\partial y}{\partial t} = 0$$

or

$$\frac{\partial q}{\partial x} + \frac{\partial y}{\partial t} = 0 \quad (10.4)$$

Where  $q$  is the discharge per unit width of channel.

Fig 10.2, which shows the forces acting on a fluid control volume, defines the parameters required for the development of the momentum equation.



**Fig 10.2 Reference diagram for the momentum equation**

The momentum equation relates net force to momentum change:

$$F - \left( F + \frac{\partial F}{\partial x} dx \right) + W \sin \theta - \tau_0 P_f dx = \rho A dx \frac{dv}{dt} \quad (10.5)$$

where the pressure force  $F = \rho g A \bar{y}$ ; the weight force  $W = \rho g ad$ ; the wall shear stress  $\tau_0 = \rho g R_h S_f$ ;  $P_f$  is the perimeter length. Equation (10.5) may therefore be expressed in terms of more basic flow parameters:

$$-\rho g \frac{\partial}{\partial x} (A \bar{y}) dx + \rho g A S_0 dx - \rho g A S_f dx = \rho A dx \frac{dv}{dt}$$

dividing across by  $\rho g dx$ :

$$-\frac{\partial}{\partial x}(A\bar{y}) + AS_o - AS_f = \frac{A}{g} \left( v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} \right) \quad (10.6)$$

For a rectangular section this simplifies to

$$-\frac{\partial y}{\partial x} + S_o - S_f = \frac{v}{g} \frac{\partial v}{\partial x} + \frac{1}{g} \frac{\partial v}{\partial t}$$

or

$$v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} + g \frac{\partial y}{\partial x} + g(S_f - S_o) = 0 \quad (10.7)$$

The terms of the momentum equation have the dimension of acceleration or force per unit mass. The first two terms on the left-hand side are the fluid acceleration terms,  $g \partial y/\partial x$  represents the pressure force component,  $gS_f$  and  $gS_o$  represent the friction and gravity force components, respectively.

The forms of the continuity and momentum equations, represented in equations (10.3) and (10.7), respectively, are known as the Saint Venant equations; they relate the dependent variables  $y$  and  $v$  to the independent space and time variables  $x$  and  $t$ , respectively.

### 10.3 Solution by the characteristics method

The same procedure as used in Chapter 6 for the solution of the corresponding pair of equations for unsteady flow in pipes, is applied here. Multiplying the continuity equation (10.3) by the factor  $\lambda$  and adding to the momentum equation (10.7):

$$\left[ \frac{\partial v}{\partial x} (\lambda y + v) + \frac{\partial v}{\partial t} \right] + \lambda \left[ \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} \left( v + \frac{g}{\lambda} \right) \right] + g(S_f - S_o) = 0 \quad (10.8)$$

This partial differential equation can be converted to a total differential equation provided that

$$\frac{dx}{dt} = \lambda y + v = v + g/\lambda$$

or  $\lambda y = g/\lambda$ . Hence

$$\lambda = \pm \left( \frac{g}{y} \right)^{0.5}$$

and

$$\frac{dx}{dt} = v \pm (gy)^{0.5} = v \pm c \quad (10.9)$$

where  $c$  is the gravity wavespeed; hence  $\lambda = \pm g/c$ . Thus eqn (10.8) can be written in the equivalent total differential form:

$$\frac{dv}{dt} + \frac{g}{c} \frac{dy}{dt} + g(S_f - S_o) = 0 \quad (10.10)$$

subject to

$$\frac{dx}{dt} = v + c; \quad (10.11)$$

and

$$\frac{dv}{dt} - \frac{g}{c} \frac{dy}{dt} + g(S_f - S_o) = 0 \quad (10.12)$$

subject to

$$\frac{dx}{dt} = v - c \quad (10.13)$$

Thus, the two partial differential equations (10.3) and (10.7) have been converted to their characteristic form i.e. linked pairs of ordinary differential equations. On integration of the latter over the time interval  $\Delta t$  we get a pair of  $C^+$  characteristic equations and a pair of  $C^-$  characteristic equations:

$$C^+ \quad \begin{aligned} v_P - v_R + g \int_{y_R}^{y_P} \frac{1}{c} dy + \int_{t_R}^{t_P} g(S_f - S_o) dt &= 0 \\ x_P - x_R &= \int_{t_R}^{t_P} (v + c) dt \end{aligned}$$

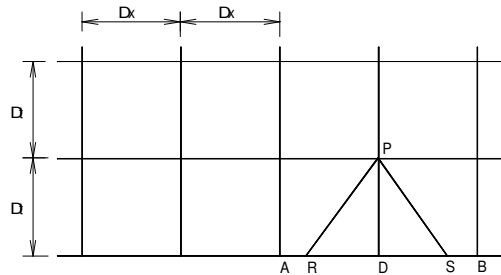
$$C^- \quad \begin{aligned} v_P - v_S - g \int_{y_S}^{y_P} \frac{1}{c} dy + \int_{t_S}^{t_P} g(S_f - S_o) dt &= 0 \\ x_P - x_S &= \int_{t_S}^{t_P} (v - c) dt \end{aligned}$$

where  $v_R$  and  $v_S$  are the interpolated values of  $v$  at  $x_R$  and  $x_S$ , respectively, as illustrated on the  $x$ - $t$  plane on Fig 10.3. The foregoing integrations may be approximated to a first order accuracy by assigning their known values to  $v$ ,  $c$  and  $S_f$ , giving the characteristic equations the following format:

$$C^+ \quad \begin{aligned} v_P - v_R + \frac{g}{c_R} (y_P - y_R) + g(S_R - S_o) \Delta t &= 0 \quad (10.14) \\ x_P - x_R &= (v_R + c_R) \Delta t \quad (10.15) \end{aligned}$$

$$C^- \quad \begin{aligned} v_P - v_S + \frac{g}{c_S} (y_P - y_S) + g(S_S - S_o) \Delta t &= 0 \quad (10.16) \\ x_P - x_S &= (v_S - c_S) \Delta t \quad (10.17) \end{aligned}$$

where  $S_R$  and  $S_S$  are the values of  $S_f$  at  $R$  and  $S$ , respectively.



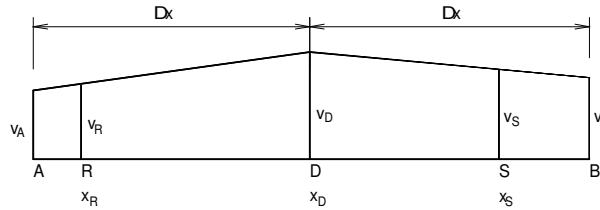
**Fig 10.3** The  $x - t$  plane

The parameter values at  $R$  are found by linear interpolation in the interval  $AD$  and the parameter values at  $S$  are found by linear interpolation in the interval  $DB$ . Referring to the interval  $AD$  in Fig 10.4:

$$x_D - x_R = \Delta t(v_R + c_R)$$

$$\frac{v_D - v_R}{v_D - v_A} = \frac{x_D - x_R}{x_D - x_A}$$

$$\frac{c_D - c_R}{c_D - c_A} = \frac{x_D - x_R}{x_D - x_A}$$



**Fig 10.4 Linear interpolation**

Replacing  $x_D$  by  $x_P$  and  $(x_D - x_A)$  by  $\Delta x$ , the following are the interpolated values at R:

$$v_R = \frac{v_D + \theta(-v_D c_A + c_D v_A)}{1 + \theta(v_D - v_A + c_D - c_A)} \quad (10.18)$$

$$c_R = \frac{c_D - v_R \theta(c_D - c_A)}{1 + \theta(c_D - c_A)} \quad (10.19)$$

$$y_R = y_D - \theta(y_D - y_A)(v_R + c_R) \quad (10.20)$$

where  $\theta = \Delta t/\Delta x$ . Interpolated values are similarly established at S on the negative characteristic side of D:

$$x_D - x_S = \Delta t(v_S - c_S)$$

$$\frac{v_D - v_S}{v_D - v_B} = \frac{x_S - x_D}{x_B - x_D}$$

$$\frac{c_D - c_S}{c_D - c_B} = \frac{x_S - x_D}{x_B - x_D}$$

Solution of these equations gives the following interpolated values at S:

$$v_S = \frac{v_D + \theta(-v_D c_B - c_D v_B)}{1 - \theta(v_D - v_B - c_D + c_B)} \quad (10.21)$$

$$c_S = \frac{c_D + v_S \theta(c_D - c_B)}{1 + \theta(c_D - c_B)} \quad (10.22)$$

$$y_S = y_D + \theta(y_D - y_B)(v_S - c_S) \quad (10.23)$$

## 10.4 Numerical computation procedure

The foregoing finite difference formulation of the characteristic form of the unsteady flow equations can be used where there are no abrupt changes in the water surface profile and where conditions are sub-critical. The computational procedure adopted is similar to that outlined in Chapter 6 for the solution of the corresponding set of pipe flow equations. The channel length is divided into  $N$  reaches, each of length  $\Delta x$ . The corresponding value of the time step  $\Delta t$  is set by the so-called Courant condition:

$$\Delta t \leq \frac{\Delta x}{|v| + c} \quad (10.24)$$

This ensures that the characteristic curves plotted on the  $x$ - $t$  plane (Fig 10.3) remain within a single  $x$ - $t$  grid. At time zero the values of  $y$  and  $v$  are known at each channel node point. Their values at internal nodes, at one time interval  $\Delta t$  later, are found by solution of equations (10.14) and (10.16) and are as follows:

$$y_P = \frac{1}{c_R + c_S} \left\{ y_S c_R + y_R c_S + c_R c_S \left[ \frac{v_R - v_S}{g} - \Delta t (S_R - S_S) \right] \right\} \quad (10.25)$$

$$v_P = v_R - \frac{g(y_P - y_R)}{c_R} - g\Delta t (S_R - S_o) \quad (10.26)$$

The updated values of  $y$  ( $y_P$ ) and  $v$  ( $v_P$ ) at the upstream end of the channel are governed by the negative characteristic equations (10.16) and (10.17) and the prevailing upstream boundary condition equation, which is typically in the form of a defined variation of either  $y$  or  $Q$  with time. Solution of equation (10.17) and the boundary condition equation yields the required values for  $v_P$  and  $y_P$ .

The new values for  $v_P$  and  $y_P$  at the downstream end of the channel are found in the same manner as their corresponding values at the upstream end, the defining equations being the positive characteristic equation (10.18) and the prevailing downstream boundary condition equation.

The foregoing analysis relates to conditions of tranquil flow only, that is, where the Froude number  $F_r$  is less than unity. As the flow depth approaches the critical value ( $F_r = 1$ ), the numerical computation becomes unstable. At critical depth,  $v = c$  and hence the negative characteristic on the  $x$ - $t$  plane becomes vertical, that is, points S and D are coincident.

### *Worked example*

The foregoing analysis has been applied to the computation of the variation of flow depth and flow rate in the following example.

The outlet sluice gate in a 4m diameter culvert is closed at a rate that linearly reduces an initial steady discharge rate of  $4 \text{ m}^3 \text{ s}^{-1}$  to zero over a period of 60 seconds. The culvert length is 500m, bottom slope is 0.0005 and the Manning  $n$ -value is 0.015. The depth of water at the upstream end of the channel remains constant at its steady flow value. The output of the computation is plotted in Fig 10.5, which shows the variations of the following parameters with time:

- upstream depth (constant at initial steady flow value of 1.257m)
- downstream depth (oscillating value)
- inflow to channel (oscillating value)
- outflow from channel (reduces to zero in 60s)

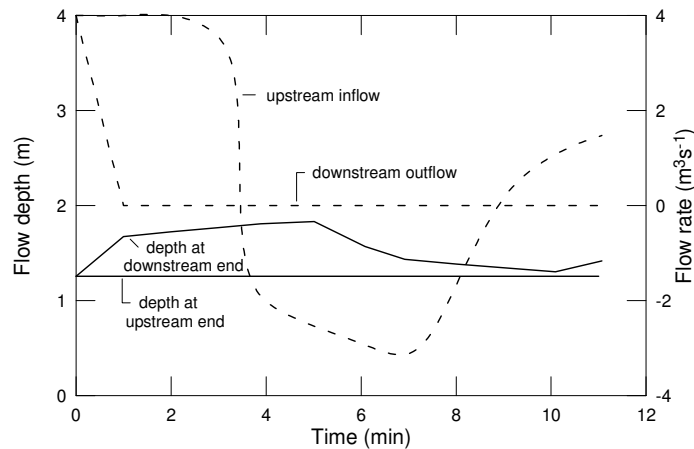


Fig 10.5 Worked example: plotted output data

### 10.5 Simplification of the St Venant equations

The Saint Venant equations can be made more amenable to solution by omitting selected terms from the momentum equation (10.2). The latter may be written in the form

$$S_o = S_f + \frac{\partial y}{\partial x} + \frac{g}{v} \frac{\partial v}{\partial x} + \frac{1}{g} \frac{\partial v}{\partial t} \quad (10.27)$$

Henderson has pointed out that the acceleration terms (3rd. and 4th. on the right-hand side of (10.27)) are usually two orders of magnitude less than the gravity ( $S_o$ ) and friction ( $S_f$ ) terms and one or two orders of magnitude less than the remaining term  $\partial y/\partial x$ . This suggests that the solution of the simplified equation obtained by dropping the acceleration terms may provide a good approximation to the solution based on the full equations. The resulting simplified momentum equation becomes

$$\frac{dy}{dx} = S_o - S_f \quad (10.28)$$

On combination with the continuity equation (10.3) the resulting unsteady open channel flow equation for a rectangular channel has the form

$$\frac{y}{v} \frac{\partial v}{\partial x} + \frac{1}{v} \frac{\partial y}{\partial t} = S_f - S_o \quad (10.29)$$

A further simplification of the momentum equation is obtained by omission of the  $dy/dx$  term (this term represents the unbalanced pressure force component), reducing the momentum equation to its steady uniform flow form:

$$S_f = S_o \quad (10.30)$$

Using the Manning expression of friction slope equation (10.30) becomes

$$S_o = \left( \frac{nQ}{AR_h^{0.67}} \right)^2 \quad (10.31)$$

and hence we can write

$$Q = f(A) \quad \text{and} \quad \frac{dQ}{dx} = \frac{dQ}{dA} \frac{dA}{dx} = F(A) \frac{dA}{dx}$$

Combining this form of simplified momentum equation with the continuity equation (10.2), where  $q_1 = 0$ , the resulting open channel unsteady flow equation becomes:

$$F(A) \frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} = 0 \quad (10.32)$$

This equation is known as the kinematic wave equation because the dynamic terms of the momentum equation have been omitted in its development. The solution of equation (10.32) is clearly of the form:

$$A = \phi \left( t - \frac{x}{F(A)} \right) \quad (10.33)$$

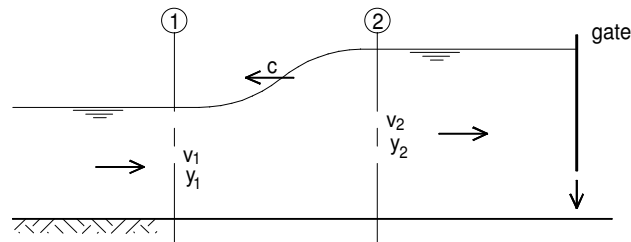
where the form of the function  $\phi$  is determined by the boundary condition for  $x=0$ .

## 10.6 Rapidly varied unsteady flow

Rapidly varied unsteady flow gives rise to a surge or wave front, which moves as a step-change in water depth along the channel. A positive surge is defined as one which leaves an increased water depth in its wake as the wave front passes, while a negative surge is one which leaves a shallower depth in its wake as the wave front passes. In the following simplified analysis of surge front movement the effect of frictional resistance is neglected.

### 10.6.1 Upstream positive surge

An upstream positive surge may be created in channel flow, for example, by the rapid closure of a gate, resulting in a step reduction in flow rate. The effect of this on the upstream side of the gate is the development of a wave front, which travels upstream, as illustrated on Fig 10.6.



**Fig 10.6** Upstream positive surge

Referring to Fig 10.6, the surge front is seen to leave in its wake an increased depth  $y_2$ , hence the description 'positive'. By superimposing a downstream velocity  $c$  on the flow system, the flow regime is converted to an equivalent steady state, in which the wave front is now stationary. Applying the continuity and momentum principles to the control volume between sections 1 and 2, under the transformed steady state conditions:

$$\text{Continuity} \quad A_1 (v_1 + c) = A_2 (v_2 + c) \quad (10.34)$$

Hence

$$v_2 = \frac{A_1 v_1 - c(A_2 - A_1)}{A_2}$$

and

$$c = \frac{Q_1 - Q_2}{A_2 - A_1}$$

Neglecting the flow friction force:

$$\text{Momentum} \quad \rho g \bar{y}_1 A_1 - \rho g \bar{y}_2 A_2 = \rho A_1 (v_1 + c)(v_2 - v_1) \quad (10.35)$$

Solving equations (10.34) and (10.35) for  $c$  and  $v_2$ :

$$c = \left[ g A_2 \frac{A_2 \bar{y}_2 - A_1 \bar{y}_1}{A_1 (A_2 - A_1)} \right]^{0.5} - v_1 \quad (10.36)$$

$$v_2 = v_1 - \left[ \frac{g (A_2 - A_1) (A_2 \bar{y}_2 - A_1 \bar{y}_1)}{A_1 A_2} \right]^{0.5} \quad (10.37)$$

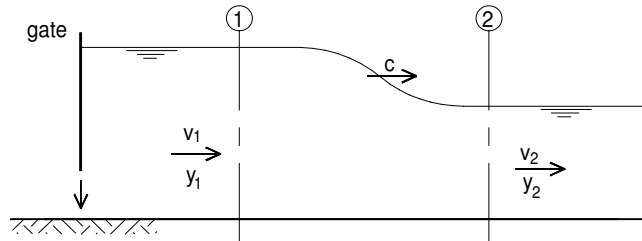
For a rectangular channel:

$$c = \left[ \frac{g y_2 (y_2 + y_1)}{2 y_1} \right]^{0.5} - v_1 \quad (10.38)$$

If  $c$  is assumed equal to zero in equation (10.38) the resulting relation between  $y_1$  and  $y_2$  is that for a hydraulic jump. Thus, the hydraulic jump can be considered to be a stationary surge. It should be noted that the continuity and momentum equations are not sufficient on their own to define the flow regime since there are three unknowns,  $c$ ,  $y_2$  and  $v_2$  (or  $Q_2$ ). One of these must therefore be known to enable computation of the remaining two parameters.

### 10.6.2 Downstream positive surge

A downstream positive surge is caused, for example, by the sudden opening of a gate, which results in an instantaneous increase in discharge and flow depth downstream of the gate, as illustrated on Fig 10.7.



**Fig 10.7** Downstream positive surge

Applying the same analytical approach as used for the analysis of the upstream positive surge, the flow regime is transformed to an equivalent steady state by superimposing a backward velocity of magnitude  $c$  on the system. Thus, referring to Fig 10.7, the continuity and momentum principles can be applied to the control volume defined by sections 1 and 2:

$$\text{Continuity} \quad A_1 (v_1 - c) = A_2 (v_2 - c) \quad (10.39)$$

Hence

$$c = \frac{Q_1 - Q_2}{A_1 - A_2} \quad (10.40)$$

Neglecting the flow friction force:

$$\text{Momentum} \quad \rho g \bar{y}_1 A_1 - \rho g \bar{y}_2 A_2 = \rho A_2 (v_2 - c)(v_2 - v_1) \quad (10.41)$$

Solving equations (10.39) and (10.41) for  $c$  and  $v_1$ :

$$c = \left[ g A_1 \frac{A_1 \bar{y}_1 - A_2 \bar{y}_2}{A_2 (A_1 - A_2)} \right]^{0.5} + v_2 \quad (10.42)$$

$$v_1 = v_2 + \left[ \frac{g (A_1 - A_2) (A_1 \bar{y}_1 - A_2 \bar{y}_2)}{A_1 A_2} \right]^{0.5} \quad (10.43)$$

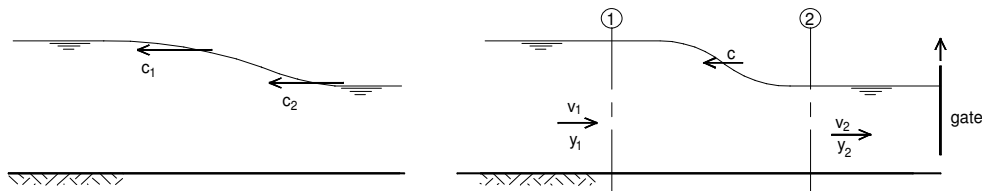
For a rectangular channel:

$$c = \left[ \frac{g y_1 (y_2 + y_1)}{2 y_2} \right]^{0.5} + v_2 \quad (10.44)$$

$$v_1 = \left[ \frac{g (y_1 y_2)^2 (y_1 + y_2)}{2 y_1 y_2} \right]^{0.5} + v_2 \quad (10.45)$$

### 10.6.3 Upstream negative surge

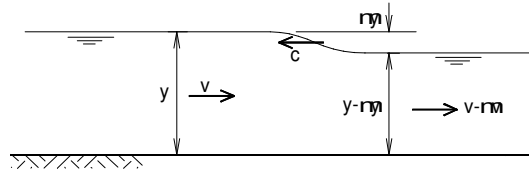
A negative surge is seen by the observer as a wave front movement which leaves a lowered water surface level in its wake. An upstream negative surge may be caused, for example, upstream of a rapidly opened gate, as illustrated on Fig 10.8. The wavefront flattens as it travels along the channel, due to the top of the wave having a greater velocity than the bottom. It is necessary, therefore, to calculate two wavespeeds, one for the wave crest and the second for the wave trough.



**Fig 10.8** Upstream negative surge

Consider a small rapid disturbance, giving rise to a small negative surge, moving upstream as illustrated on Fig 10.9. Applying the continuity and momentum principles as before:

$$\text{Continuity} \quad (v+c)y = (y-\partial y)(v-\partial v+c) \quad (10.46)$$



**Fig 10.9 Negative surge propagation**

$$\text{Momentum} \quad \rho g \left( \frac{y^2}{2} - \frac{(y - \partial y)^2}{2} \right) = \rho y (v + c)(-\partial v) \quad (10.47)$$

from (10.46)  $\partial y = -y \partial v / (v + c)$ ; from eqn(10.47)  $\partial y = -\partial v (v + c) / g$ .

Hence

$$c = \sqrt{gy} - v \quad (10.48)$$

and also

$$\partial y = -\frac{\partial v}{g} \sqrt{gy}$$

which, as  $\partial y$  approaches zero, can be written

$$\frac{dy}{\sqrt{y}} = -\frac{dv}{\sqrt{g}}$$

Integrating for a wave of finite height

$$v = -2\sqrt{gy} + \text{constant} \quad (10.49)$$

For the upstream negative surge, illustrated on Fig 10.8, we have the known boundary condition,  $v = v_1$  when  $y = y_1$ ; using these values in equation (10.49), the integration constant is found to be  $2\sqrt{gy_1} + v_1$ . Hence, from eqn (10.49) :

$$v_2 = 2\sqrt{gy_1} - 2\sqrt{gy_2} + v_1 \quad (10.50)$$

and from equation (10.48)

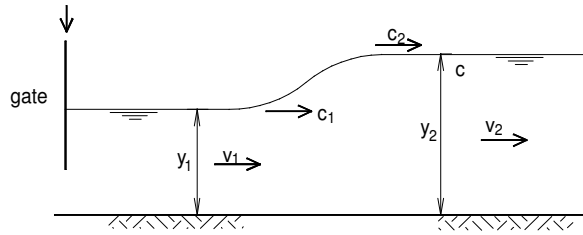
$$c_2 = \sqrt{gy_2} - v_2$$

Hence from (10.50)

$$c_2 = 3\sqrt{gy_2} - 2\sqrt{gy_1} - v_1 \quad (10.51)$$

#### 10.6.4 Downstream negative surge

A downstream negative surge is propagated downstream of a rapidly closed gate, for example, as illustrated on Fig 10.10.



**Fig 10.10 Downstream negative surge**

Using the same analytical procedure as applied in the case of the upstream negative surge, the wave front velocity  $c$  can be shown, in the case of a downstream negative surge, to be:

$$c = \sqrt{gy} + v \quad (10.52)$$

and the velocity is

$$v = 2\sqrt{gy} - 2\sqrt{gy_2} + v_2 \quad (10.53)$$

Hence the values of  $v_1$  and  $c_1$  are found to be

$$v_1 = 2\sqrt{gy_1} - 2\sqrt{gy_2} + v_2 \quad (10.54)$$

$$c_1 = 3\sqrt{gy_1} - 2\sqrt{gy_2} + v_2 \quad (10.55)$$

### Related reading

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